Covering and distortion theorems for spirallike functions with respect to a boundary point[*†]

Mark Elin

Department of Mathematics, ORT Braude College,

P.O. Box 78, Karmiel 21982, ISRAEL

e-mail: mark.elin@gmail.com

Abstract

In this note we obtain some distortion results for spirallike functions with respect to a boundary point. In particular, we find the maximal domain covered by all spirallike functions of order β.

0 Introduction

Although the classes of starlike and spirallike functions (taking zero at zero) were objects of interest during about the last hundred years, the study of functions having similar geometric properties with respect to a boundary point was begun only in the 1980's.

A breakthrough in this matter is due to M. S. Robertson [13], who defined the class of those univalent holomorphic functions $f \in \mathrm{Hol}(\Delta, \mathbb{C})$ on the open unit disk Δ satisfying $f(0) = 1$ such that $f(\Delta)$ is starlike with respect to the boundary point $f(1) := \lim\limits_{r\to 1^-} f(r) = 0$ and $f(\Delta)$ lies in a half-plane. Different characterizations of the class of starlike functions with respect to a boundary point were obtained in [12, 18, 5, 9, 11]; properties of these functions were considered in numerous works (see, for example, [1, 19]). Various distortion

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results for starlike functions with respect to a boundary point were proved in [15, 3, 4].

It seems that except for [6], the paper [2] was the first where spirallike functions with respect to a boundary point were studied systematically.

Definition 0.1 *A univalent function* $f \in \mathrm{Hol}(\Delta, \mathbb{C})$ *normalized by* $f(0) = 1$ *and* $f(1) := \lim\limits_{r\to 1^-} f(r) = 0$ *is called spirallike with respect to a boundary point if there is a number* $\mu \in \mathbb{C},\ \mathrm{Re}\,\mu > 0,$ *such that for any point* $w \in f(\Delta)$ *the curve* $\{e^{-t\mu}w,\ t \geq 0\}$ *is contained in* $f(\Delta)$*.*

If, in particular, we also have $\mu \in \mathbb{R}$*, then* f *is called starlike with respect to a boundary point.*

The following assertion was proved in [2].

Theorem 0.1 *Let* $f \in \mathrm{Hol}(\Delta, \mathbb{C}),\ f(0) = 1,\ f(1) = 0,$ *be a spirallike function with respect to a boundary point. Then there exists a number* $\mu \in \Omega := \{\lambda \in \mathbb{C} :\ |\lambda - 1| \leq 1,\ \lambda \neq 0\}$ *such that*

$$\mathrm{Re}\left(\frac{2}{\mu}\,\frac{zf'(z)}{f(z)} + \frac{1+z}{1-z}\right) > 0. \tag{0.1}$$

Conversely, if a univalent function $f,\ f(0) = 1,\ f(1) = 0,$ *satisfies (0.1) for some* $\mu \in \Omega$*, then* f *is a spirallike function with respect to a boundary point.*

Further, this class was investigated in [7, 17, 10]. In particular, combining results of [7] and [2] one can formulate the following geometric characterization of images of spirallike functions with respect to a boundary point.

Theorem 0.2 *Let* $f \in \mathrm{Hol}(\Delta, \mathbb{C}),\ f(0) = 1,\ f(1) = 0,$ *be a spirallike function with respect to a boundary point. Then*

(i) the following two limits exist finitely:

$$\lim_{r\to 1^-} \frac{f'(r)(r-1)}{f(r)} = \nu \tag{0.2}$$

with $\nu \in \Omega$*, and*

$$\lim_{r\to 1^-} \arg f^{1/\nu}(r) = a \tag{0.3}$$

with $|a| < \dfrac{\pi}{2}$*;*

(ii) inequality (0.1) holds for $\mu = \nu$ defined by (0.2); moreover, any number μ for which inequality (0.1) holds must satisfy $\mu = R\nu,\ R \geq 1$;

(iii) the minimal spiral wedge that contains the image $f(\Delta)$ is bounded by two spirals:

$$w = e^{-\nu t} w_+ \text{ and } w = e^{-\nu t} w_-,\ t \in \mathbb{R},$$

where $w_\pm = \exp\Big(i\nu(a \pm \frac{\pi}{2})\Big)$.

Note that the spiral wedge described in (iii) is the image of the spirallike function with respect to a boundary point $h_{\nu,a}(z) := \left(\frac{1-z}{1+e^{-2ia}z}\right)^\nu$, so f is subordinate to $h_{\nu,a}$ (see Definition 1.1 below).

In the present work we establish a number of distortion results and sharp covering theorems for some subclasses of univalent functions spirallike with respect to a boundary point. Namely, we consider the following classes of functions.

Definition 0.2 *We say that a univalent function $f \in \mathrm{Hol}(\Delta, \mathbb{C})$, $f(0) = 1$, belongs to the class $\mathrm{G}(\mu,\beta)$ (is μ-spirallike of order β with respect to a boundary point), where $\mu \in \Omega := \{\lambda \in \mathbb{C} :\ |\lambda - 1| \leq 0,\ \lambda \neq 0\}$ and $\beta \in [0, 1)$, if f satisfies the inequality:*

$$\operatorname{Re}\left(\frac{2}{\mu} \cdot \frac{zf'(z)}{f(z)} + \frac{1+z}{1-z}\right) > \beta, \quad z \in \Delta. \tag{0.4}$$

1 Integral representation for the class $\mathrm{G}(\mu,\beta)$

In this section we prove an integral representation for functions of the class $\mathrm{G}(\mu,\beta)$ and deduce some its consequences.

Theorem 1.1 *$f \in \mathrm{G}(\mu,\beta)$ if and only if f admits the following representation:*

$$f(z) = (1-z)^\mu \exp\left[-\mu(1-\beta) \oint \ln(1 - z\bar{\zeta}) d\sigma(\zeta)\right], \tag{1.1}$$

where σ is a probability measure on the unit circle $|\zeta| = 1$.

Proof. First, suppose that the function f is represented by (1.1). Differentiating (1.1) and doing a simple calculation we check that inequality (0.4) holds.

Suppose now that $f \in \mathrm{G}(\mu,\beta)$. Consider the function

$$h(z) = \frac{1}{1-\beta}\left[\frac{2}{\mu}\frac{zf'(z)}{f(z)} + \frac{1+z}{1-z} - \beta\right],$$

which obviously belongs to the Carathéodory class and thus can be represented by the Riesz–Herglotz formula:

$$h(z) = \oint \frac{1+z\bar{\zeta}}{1-z\bar{\zeta}}\, d\sigma(\zeta),$$

where σ is a probability measure. This implies that

$$\frac{f'(z)}{\mu f(z)} = \oint_{|\zeta|=1}\left[\frac{(1-\beta)\bar{\zeta}}{1-z\bar{\zeta}} - \frac{1}{1-z}\right] d\sigma(\zeta).$$

Integrating both sides of this equations we obtain (1.1). ■

Remark. It follows by Theorem 0.1, Theorem 1.1 and Definition 0.2 that any function f spirallike with respect to a boundary point has the integral representation

$$f(z) = (1-z)^{\mu}\exp\left[-\mu\oint \ln(1-z\bar{\zeta})\, d\sigma(\zeta)\right]$$

for some $\mu \in \Omega$ and a probability measure $d\sigma$. Define another measure $d\widetilde{\sigma}$ such that $d\sigma = \beta d\lambda + (1-\beta)d\widetilde{\sigma}$, where $d\lambda$ is the normalized Lebesgue measure on the unit circle $\partial\Delta$. Obviously, $\int\limits_{\partial\Delta} d\widetilde{\sigma} = 1$. Since the integral of the antiholomorphic function $\log(1-z\bar{\zeta})$ with respect to the Lebesgue measure $d\lambda$ is zero, we have that f belongs to the class $\mathrm{G}(\mu,\beta)$ if and only if the measure $d\widetilde{\sigma} = \frac{1}{1-\beta}\left(d\sigma - \beta d\lambda\right)$ is positive.

The following assertion is an immediate consequence of Theorem 1.1.

Corollary 1.1 *Let $\mu_1,\mu_2 \in \Omega$ and $\beta_1,\beta_2 \in [0,1)$. Let $f \in \mathrm{Hol}(\Delta,\mathbb{C})$ be univalent. Then $f \in G(\mu_1,\beta_1)$ if and only if the function*

$$\widetilde{f}(z) = (1-z)^{\mu_2(\beta_2-\beta_1)}(f(z))^{\frac{\mu_2(1-\beta_2)}{\mu_1(1-\beta_1)}}$$

belongs to $G(\mu_2,\beta_2)$. In particular, $f \in \mathrm{G}(\mu,\beta)$ if and only if $\widetilde{f}(z) = (f(z))^{\frac{1}{\mu}}$ belongs to $G(1,\beta)$.

Corollary 1.2 *Let $\mu_1, \mu_2 \in \Omega$, $\mu_1 = r\mu_2$ with $r \le 1$, and let $\beta_2 \le r\beta_1$. Then $G(\mu_1, \beta_1) \subset G(\mu_2, \beta_2)$.*

Proof. Let $f \in G(\mu_1, \beta_1)$. We have

$$\begin{aligned} f(z) &= (1-z)^{\mu_1} \exp\left[-\mu_1(1-\beta_1) \oint_{|\zeta|=1} \ln(1-z\bar{\zeta}) d\sigma(\zeta)\right] \\ &= (1-z)^{\mu_2} \exp\left[-\mu_2(1-r\beta_1) \oint_{|\zeta|=1} \ln(1-z\bar{\zeta}) d\widetilde{\sigma}(\zeta)\right], \end{aligned}$$

where $d\widetilde{\sigma}$ is a probability measure on the unit circle defined by

$$d\widetilde{\sigma}(\zeta) = \frac{r(1-\beta_1)}{1-r\beta_1} d\sigma(\zeta) + \frac{1-r}{1-r\beta_1} \delta(\zeta),$$

with δ the Dirac measure at the point $\zeta = 1$. By Theorem 1.1, f satisfies the inequality

$$\operatorname{Re}\left(\frac{2}{\mu_2} \cdot \frac{zf'(z)}{f(z)} + \frac{1+z}{1-z}\right) > r\beta_1 \ge \beta_2.$$

Therefore, $f \in G(\mu_2, \beta_2)$. ■

Corollary 1.3 *The set*

$$\bigcup_{n=1}^{\infty} \left\{ \left(\frac{1-z}{\prod_{j=1}^{n} (1-z\bar{\zeta}_j)^{\lambda_j}} \right)^{\mu} : \lambda_j \ge 0,\ \sum_{j=1}^{n} \lambda_j = 1-\beta,\ |\zeta_j| = 1 \right\}$$

is dense in $\mathrm{G}(\mu, \beta)$ *in the topology of uniform convergence on compact subsets of Δ.*

Proof. Replacing the integral in (1.1) by approximating sums, we have for any $f \in G(\mu, \beta)$:

$$\begin{aligned} f(z) &= \lim_{n\to\infty} (1-z)^{\mu} \exp\left[-\mu(1-\beta) \sum_{j=1}^{n} \ln(1-z\bar{\zeta}_j)\sigma_j\right] \\ &= \lim_{n\to\infty} (1-z)^{\mu} \prod_{j=1}^{n} (1-z\bar{\zeta}_j)^{-\mu(1-\beta)\sigma_j}, \end{aligned}$$

with $\sum_{j=1}^{n} \sigma_j = 1$. Putting $\lambda_j = (1-\beta)\sigma_j$, we complete the proof. ■

To formulate our next result we need the notion of subordination.

Definition 1.1 *A function* $s_1 \in \mathrm{Hol}(\Delta, \mathbb{C})$ *is said to be subordinate to a function* $s_2 \in \mathrm{Hol}(\Delta, \mathbb{C})$ *(*$s_1 \prec s_2$*) if there exists a holomorphic function* ω *on* Δ *with* $|\omega(z)| \leq |z|$, $z \in \Delta$*, such that* $s_1 = s_2 \circ \omega$*.*

Corollary 1.4 *Let* $f \in \mathrm{G}(\mu, \beta)$*. Then* $\dfrac{f(z)}{(1-z)^\mu} \prec \dfrac{1}{(1-z)^{\mu(1-\beta)}}$.

Proof. Since the function $\log(1-z)$ is convex, for any probability measure $d\sigma$ there exists an analytic function $\omega : \Delta \mapsto \Delta$ with $\omega(0) = 0$ such that

$$\oint \ln(1 - z\bar{\zeta}) d\sigma(\zeta) = \log(1 - \omega(z)).$$

Using Theorem 1.1 we obtain that

$$\frac{f(z)}{(1-z)^\mu} = \exp\left[-\mu(1-\beta) \oint \ln(1 - z\bar{\zeta}) d\sigma(\zeta)\right] = \frac{1}{(1-\omega(z))^{\mu(1-\beta)}} ,$$

which proves our assertion. ■

It turns out that for the classes $\mathrm{G}(\mu, \beta)$ one can improve the result of Theorem 0.2 (iii).

Corollary 1.5 *Let* $f \in \mathrm{G}(\mu, \beta)$*. Then* $f \prec h_{\nu,a}(z) := \left(\frac{1-z}{1+e^{-2ia}z}\right)^\nu$*, where* $\frac{\nu}{\mu}$ *is a real number satisfying* $\beta \leq \frac{\nu}{\mu} \leq 1$ *and* $|a| < \frac{\pi}{2}\frac{\mu}{\nu}(1-\beta)$*. Consequently,* $|a| < \frac{\pi}{2} \cdot \min\left(1, \frac{1-\beta}{\beta}\right)$.

Proof. Using representation (1.1) one can estimate the numbers ν and a from Theorem 0.2 (i). Specifically,

$$\frac{f'(r)(r-1)}{\mu f(r)} = 1 - (1-\beta)(1-r) \int_{\partial\Delta} \frac{\bar{\zeta}}{1 - r\bar{\zeta}} d\sigma(\zeta).$$

Thus $\frac{\nu}{\mu} = \lim\limits_{r \to 1^-} \frac{f'(r)(r-1)}{\mu f(r)} \geq \beta$.

Further,

$$|a| = \lim_{r \to 1^-} = \frac{\mu}{\nu}(1-\beta) \left| \int_{\partial\Delta} \arg(1 - r\bar{\zeta}) d\sigma(\zeta) \right|.$$

Thus the assertion follows. ■

Remark. For the case when the number ν in (0.2) (consequently, μ in (0.4)) is real, Theorem 0.2 asserts that the image of a function f is contained in the wedge of angle $\nu\pi$ with the midline $\arg w = \nu a$. The last Corollary implies that if $f \in \mathrm{G}(\mu, \beta)$ then the angle can not be less than $\pi\mu\beta$ and the argument of the midline can not be greater than $\frac{\pi}{2}\mu(1-\beta)$.

2 Estimates for f and f'

In this section we establish distortion theorems for classes $\mathrm{G}(\mu,\beta)$. In particular, given a point $z\in\Delta$, we find the set of values for some functionals on this classes. For the class of Robertson (the class of starlike functions with respect to a boundary point having image in a half-plane) similar results can be found in [15] (see also [3, 18, 4]).

Theorem 2.1 *For each fixed $z\in\Delta$ we have*

$$(i)\qquad \left\{\frac{1-z}{f(z)^{1/\mu}}:\ f\in\mathrm{G}(\mu,\beta)\right\}=\left\{(1+\lambda z)^{1-\beta},\ |\lambda|\le 1\right\} \tag{2.1}$$

and

$$(ii)\qquad \left\{\frac{f'(z)}{\mu f(z)}+\frac{1}{1-z}:\ f\in\mathrm{G}(\mu,\beta)\right\} \tag{2.2}$$
$$=\left\{w:\ \left|w-\frac{(1-\beta)\bar z}{1-|z|^2}\right|\le\frac{1-\beta}{1-|z|^2}\right\}.$$

Furthermore, if $f\in\mathrm{G}(\mu,\beta)$ and $z\in\Delta,\ z\neq 0$, one of the relations

$$\frac{1-z}{f(z)^{1/\mu}}\in\left\{(1+\lambda z)^{1-\beta},\ |\lambda|=1\right\}$$

and

$$\frac{f'(z)}{\mu f(z)}+\frac{1}{1-z}\in\left\{w:\ \left|w-\frac{(1-\beta)\bar z}{1-|z|^2}\right|=\frac{1-\beta}{1-|z|^2}\right\}$$

holds only if

$$f(z)=\frac{(1-z)^{\mu}}{(1-z\bar\xi)^{(1-\beta)\mu}},\qquad \xi\in\partial\Delta. \tag{2.3}$$

Proof. (i) By Theorem 1.1

$$\log\left[\frac{1-z}{f(z)^{1/\mu}}\right]=(1-\beta)\oint_{\partial\Delta}\log(1-z\bar\zeta)d\sigma(\zeta). \tag{2.4}$$

By the Carathéodory principle

$$\left\{\log\left(\frac{1-z}{f(z)^{1/\mu}}\right):\ f\in\mathrm{G}(\mu,\beta)\right\}=\mathrm{Conv}\left\{(1-\beta)\log(1-z\bar\zeta),\ \zeta\in\partial\Delta\right\},$$

where Conv denotes the convex hull.

Since for each $z \in \Delta$ the function $g(w) := (1-\beta)\log(1-zw)$ maps Δ onto a strictly convex domain, this formula and (2.4) imply that the value $\log\left[\frac{1-z}{f(z)^{1/\mu}}\right]$ belongs to the image $g(\Delta)$ except for the case when the measure $d\sigma(\zeta)$ is the Dirac δ-function at some boundary point ξ. Hence the assertion follows.

(ii) Once again using representation (1.1) we get

$$\frac{f'(z)}{\mu f(z)} + \frac{1}{1-z} = (1-\beta)\oint_{\partial\Delta} \frac{\bar{\zeta}}{1-z\bar{\zeta}}\, d\sigma(\zeta).$$

For each fixed $z \in \Delta$ the function $g : \partial\Delta \mapsto \mathbb{C}$ defined by $g(\zeta) := \frac{\bar{\zeta}}{1-z\bar{\zeta}}$ maps the unit circle onto the circle $\left\{ w : \left| w - \frac{\bar{z}}{1-|z|^2} \right| = \frac{1}{1-|z|^2} \right\}$. Therefore, (2.2) follows from the Carathéodory principle. ■

The following two assertions are immediate consequences of Theorem 2.1 (i).

Corollary 2.1 $\bigcup_{z\in\Delta} \left\{ \frac{1-z}{f(z)^{1/\mu}} : \ f \in \mathrm{G}(\mu,\beta) \right\} = \left\{ (1+\lambda)^{1-\beta}, \ |\lambda| \le 1 \right\}.$

Corollary 2.2 *For each $z \in \Delta$ and $f \in \mathrm{G}(\mu,\beta)$,*

$$(1-|z|)^{1-\beta} \le \left| \frac{1-z}{f(z)^{1/\mu}} \right| \le (1+|z|)^{1-\beta}$$

and

$$\left| \arg\left(\frac{1-z}{f(z)^{1/\mu}} \right) \right| \le (1-\beta)\arcsin|z|,$$

where for $z \neq 0$ equality is achieved only for the functions (2.3) at the points $z = \pm|z|\xi$ and $z = |z|\xi e^{\pm i \arccos|z|}$, respectively. In particular, if μ is real, then

$$\frac{|1-z|^{\mu}}{(1+|z|)^{\mu(1-\beta)}} \le |f(z)| \le \frac{|1-z|^{\mu}}{(1-|z|)^{\mu(1-\beta)}}.$$

In the case when $\mu \in (0,2]$ (i.e., $f \in \mathrm{G}(\mu,\beta)$ is, in fact, starlike) one obtains the following estimate for f'.

Corollary 2.3 *For each* $z \in \Delta$ *and* $f \in \mathrm{G}(\mu,\beta),\ \mu \in (0,2],\ \beta \in [0,1)$,

$$\frac{\mu|1-z|^{\mu}}{(1-|z|^2)(1+|z|)^{\mu(1-\beta)}}\left(\left|\frac{1-\bar{z}}{1-z}+\beta\bar{z}\right|-1+\beta\right) \le |f'(z)| \le$$
$$\le \frac{\mu|1-z|^{\mu}}{(1-|z|^2)(1-|z|)^{\mu(1-\beta)}}\left(\left|\frac{1-\bar{z}}{1-z}+\beta\bar{z}\right|+1-\beta\right).$$

In particular,

$$|f'(z)| \le \frac{2\mu|1-z|^{\mu}}{(1-|z|^2)(1-|z|)^{\mu(1-\beta)}}\,.$$

The proof follows from Theorem 2.1 (ii) and Corollary 2.2.

3 Covering theorem

In this section we prove a covering theorem for classes $\mathrm{G}(\mu,\beta)$ of spirallike functions with respect to a boundary point. Note that if for some class of functions there exists a domain D such that $D \subset f(\Delta)$ for each function f of the class, and the function $f_0 \in \mathrm{Hol}(\Delta,\mathbb{C})$ maps the open unit disk onto D conformly, then the corresponding covering result can be expressed as the subordination $f_0 \prec f$.

Theorem 3.1 *Let* $f \in \mathrm{G}(\mu,\beta)$. *Denote* $f_0(z) = (1-z)^{\mu\beta}$. *Then* $f_0 \prec f$. *This subordination is sharp since* $f_0 \in \mathrm{G}(\mu,\beta)$.

Proof. By Corollary 1.1 it is sufficient to prove this assertion for the case $\mu = 1$. First we suppose that $f \in \mathrm{G}(1,\beta)$ has the form

$$f(z) = \frac{1-z}{\prod_{j=1}^{n}(1-z\overline{\zeta_j})^{(1-\beta)\sigma_j}},\ \sum_{j=1}^{n}\lambda_j = 1-\beta,\ |\zeta_j| = 1.$$

Let $z_0 = e^{i\phi} \in \partial\Delta,\ z_0 \ne \zeta_1,\zeta_2,\ldots,\zeta_n$. Consider the two univalent convex functions

$$g_0(z) = \log f_0(z) = \beta\log(1-z)$$

and

$$g(z) = \log(1-z_0) - (1-\beta)\log(1-z_0 z).$$

It is easy to see that $g_0(z_0) = g(1)$, i.e., this is their common boundary point. Furthermore,

$$zg_0'(z)|_{z=z_0} = \frac{-\beta z_0}{1-z_0} \quad \text{and} \quad zg'(z)|_{z=1} = \frac{(1-\beta)z_0}{1-z_0}\,,$$

i.e., the exterior normal vectors to the images $g_0(\Delta)$ and $g(\Delta)$ at this common point have reverse directions. Because these functions are convex, $g_0(\Delta) \bigcap g(\Delta) = \emptyset$. Thus,

$$\log f(z_0) = \sum_{j=1}^{n} \sigma_j \left[\log(1-z_0) - (1-\beta)\log(1-z_0\overline{\zeta_j})\right]$$

belongs to $\overline{g(\Delta)}$ and does not belong to $g_0(\Delta) = \log f_0(\Delta)$. Since the point $f(z_0)$ is an arbitrary finite boundary point of the image $f(\Delta)$, we have proved that the boundary $\partial f(\Delta)$ does not intersect $f_0(\Delta)$. This fact implies that f_0 is subordinate to any function $f \in \mathrm{G}(1,\beta)$ that has the form described above.

In light of Corollary 1.3 and by the Carathéodory Kernel Convergence Theorem we deduce the assertion of the theorem. ■

Example. Consider the function

$$f(z) = \frac{1-z}{(1-0.9z-0.4iz)^{0.2}(1-0.9z+0.4iz)^{0.2}}\,.$$

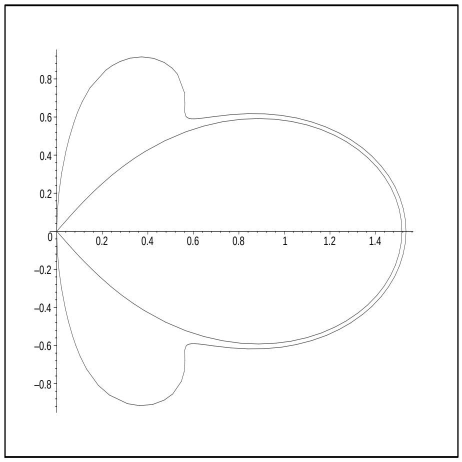
A simple calculation shows that $f \in \mathrm{G}(\mu,\beta)$ with $\mu = 1$ and $\beta = 0.6$. Hence by Theorem 3.1, its image $f(\Delta)$ contains the image $f_0(\Delta)$, where $f_0(z) = (1-z)^{0.6}$ (see Figure).

Corollary 3.1 *If $f \in \mathrm{G}(\mu,\beta)$ with $\mu \in (0,2]$ and $\beta \in (0,1)$ then the image of f covers the disk $\{w : |w-1| < r(\mu\beta)\}$, where*

$$r(s) = \begin{cases} \sqrt{1+2^{2s}-2^{s+1}}, & \text{when } s \in (0,1] \\ 1, & \text{when } s \in (1,2] \end{cases}\,.$$

Proof. It suffices to find the distance between the point $z = 1$ and the curve $\{(1-z)^s,\ z \in \partial\Delta\}$, where $s = \mu\beta$. Indeed, setting $z = -e^{it},\ t \in (-\pi,\pi)$, consider the function

$$a_s(t) = \left|(1+e^{it})^s - 1\right|^2 = \left(2\cos\frac{t}{2}\right)^{2s} + 1 - 2\left(2\cos\frac{t}{2}\right)^{s}\cos\frac{st}{2}$$



This function is even in t, a_s increases on the segment $0 \le t \le \pi$ when $s < 1$, and a_s decreases on the same segment when $s > 1$. So,

$$\min_{-\pi<t<\pi} a_s(t) = \begin{cases} a_s(0), & \text{when } s \in (0,1] \\ a_s(\pi), & \text{when } s \in (1,2] \end{cases} .$$

Calculating these values we complete the proof. ■

Remark. In [3] Chen and Owa have proved a covering theorem for starlike functions with respect to a boundary point. Using our notation their result can be quoted as follows: *If $f \in \mathrm{G}(\mu,\beta)$ for some $\mu,\beta \in [0,1]$ then its image covers the disk* $\left\{w : |w-1| < \frac{\mu\beta}{4}\right\}$. The radius of the covered disk which was found in the last Corollary is considerably larger than one due to Chen and Owa. For example, if $f \in \mathrm{G}(\mu,\beta)$ with $\mu\beta = 1$, then by the Theorem of Chen and Owa $f(\Delta)$ covers the disk of radius 1/4; our result asserts that the disk covered is of radius 1.

The following assertion connects the classes $\mathrm{G}(\mu,\beta)$ with spirallike functions with respect to an interior point.

Proposition 3.1 *Let $\beta \in [0,1)$, and let a complex number $\mu = r\,e^{i\phi}$ belong to $\Omega = \{\lambda : \ |\lambda - 1| \le 1, \ \lambda \ne 0\}$. Then a univalent function f, $f(0) = 1$, belongs to the class $\mathrm{G}(\mu,\beta)$ if and only if the function s defined by*

$$s(z) := \frac{zf(z)}{(1-z)^{\mu}} \tag{3.1}$$

is ϕ-spirallike of order $\cos\phi - \dfrac{r(1-\beta)}{2}$, *i.e., s satisfies the condition*

$$\operatorname{Re}\left(e^{-i\phi}\frac{zs'(z)}{s(z)}\right) > \cos\phi - \frac{r(1-\beta)}{2}\,, \quad z\in\Delta. \tag{3.2}$$

Remark. Note that the relation $\mu = r\,e^{i\phi} \in \Omega$ implies that $\cos\phi - \frac{r(1-\beta)}{2} \geq 0$, and then by a result of Špaček (see [8]) the function s is univalent in Δ.

Proof. Let a function s be defined by (3.1). Then

$$\begin{aligned} e^{-i\phi}\frac{zs'(z)}{s(z)} &= \frac{r}{\mu}\left(1+\frac{zf'(z)}{f(z)}+\frac{\mu z}{1-z}\right) \\ &= \frac{r}{2}\left(\frac{2zf'(z)}{\mu f(z)}+\frac{1+z}{1-z}-\beta\right)+e^{-i\phi}-\frac{r(1-\beta)}{2}\,. \end{aligned}$$

This equality implies the assertion. ■

Exactly, as in [2], using Proposition 3.1 and results of Ruscheweyh [14], one can conclude the following.

Corollary 3.2 *Let $f:\Delta\mapsto\mathbb{C}$ be a holomorphic function and $f(0)=1$. Let $\mu = re^{i\phi}\in\Omega$ and $\beta\in[0,1)$. Then $f\in \mathrm{G}(\mu,\beta)$ if and only if one of the following conditions holds:*

(a) for all $u,v\in\overline{\Delta}$

$$\left(\frac{1-uz}{1-vz}\right)^{\mu}\frac{f(vz)}{f(uz)} \prec \left(\frac{1-uz}{1-vz}\right)^{(1-\beta)\mu};$$

(b) for all $t\in(0,2\cos\phi)$

$$\left|\frac{f(z(1-e^{-i\phi}t))}{f(z)}\right| \leq \left|\left(\frac{1-z(1-e^{-i\phi}t)}{1-z}\right)^{\mu}\right|\cdot\left(1-\frac{t}{2\cos\phi}\right)^{-\operatorname{Re}\mu(1-\beta)}.$$

(We omit the proof since it repeats one in [2].)

Setting $u=0$ and $v=1$ in Corollary 3.2, we obtain an alternative proof of Corollary 1.4 above.

The next assertion follows from Theorem 3.1 and Proposition 3.1.

Corollary 3.3 *Let* $\phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\alpha < \cos\phi$, *and* $\beta \in \left(0, \frac{\alpha}{\cos\phi}\right]$. *Denote* $\mu = e^{i\phi}\frac{2(\cos\phi - \alpha)}{1-\beta}$. *For each univalent function* s *normalized by* $s(0) = 0$, $s'(0) = 1$ *and satisfying* $\operatorname{Re}\left(e^{-i\phi}\dfrac{zs'(z)}{s(z)}\right) > \alpha$, *the image of the function*

$$1 - (1-z)^{\frac{1}{\beta}} \left(\frac{s(z)}{z}\right)^{\frac{1}{\mu\beta}}$$

covers the open unit disk, and, consequently, the image of the function

$$\frac{(1-z)^{\frac{1}{\beta}} \left(\frac{s(z)}{z}\right)^{\frac{1}{\mu\beta}}}{2 - (1-z)^{\frac{1}{\beta}} \left(\frac{s(z)}{z}\right)^{\frac{1}{\mu\beta}}}$$

covers the right half-plane.

In particular, setting $\phi = 0$, $\alpha = \beta = \frac{1}{2}$, we get that for each function s starlike of order $\frac{1}{2}$ the image of the function $1 - (1-z)^2 \left(\frac{s(z)}{z}\right)$ contains the open unit disk, and the image of the function $\dfrac{(1-z)^2 s(z)}{2z - (1-z)^2 s(z)}$ contains the right half-plane.

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